

Problem 1. Let $f(x) = \sqrt{9-x}$ and $g(x) = \sqrt{x^2-25}$. Find the domains of $m(x) = f(g(x))$

Solution: The number x is in the domain of $f(g(x))$ when

(1) x is in the domain of $g(x)$

and

(2) $g(x)$ is in the domain of f .

Now x is in the domain of $g(x) = \sqrt{x^2-25}$ when

$$x^2 - 25 \geq 0,$$

or when

$$x^2 \geq 25.$$

Since the function $w(x) = \sqrt{x}$ is an increasing function, taking the square root of both sides preserves the inequality and

$$\sqrt{x^2} \geq \sqrt{25}$$

and

$$|x| \geq 5.$$

So the domain of g is

$$D_g = \{x \leq -5 \text{ or } x \geq 5\}.$$

Also, x is in the domain of $f(x) = \sqrt{9-x}$ when

$$9 - x \geq 0,$$

or when

$$x \leq 9.$$

So the domain of f is

$$D_f = \{x \mid x \leq 9\}.$$

So $g(x)$ is in the domain of f when

$$g(x) \leq 9$$

or when

$$\sqrt{x^2 - 25} \leq 9.$$

Since the function $w(x) = x^2$ is increasing for $x \geq 0$ and $\sqrt{x^2 - 9} \geq 0$, we can square both sides and preserve the inequality. So

$$x^2 - 25 \leq 81$$

and

$$x^2 \leq 106$$

So

$$|x| \leq \sqrt{106}.$$

and

$$-\sqrt{106} \leq x \leq \sqrt{106}.$$

So the domain of $m(x) = f(g(x))$ is

$$\begin{aligned} D_m &= D_g \cap \{x \mid -\sqrt{106} \leq x \leq \sqrt{106}\} \\ &= \{x \leq -5 \text{ or } x \geq 5\} \cap \{x \mid -\sqrt{106} \leq x \leq \sqrt{106}\} \\ &= \{x \mid -\sqrt{106} \leq x \leq -5 \text{ or } 5 \leq x \leq \sqrt{106}\}. \end{aligned}$$

Conclusion: The domain of $m(x) = f(g(x))$ is

$$D_m = \{x \mid -\sqrt{106} \leq x \leq -5 \text{ or } 5 \leq x \leq \sqrt{106}\}.$$

NOTES: (a) To understand the final step, you should draw the two sets

$$\{x \leq -5 \text{ or } x \geq 5\}$$

and

$$\{x \mid -\sqrt{106} \leq x \leq \sqrt{106}\}$$

on a number line and see where they intersect. **I did not show it here, but this needs to be part of your solution.**

(b) When finding the domain of a composition of functions, you should **not** compute their composition. It is often difficult and sometimes misleading.

For example, suppose $f(x) = \sqrt{x-5}$ and $g(x) = x^2$, and that we wish to find the domain of $g(f(x))$. If we compute the composition, we get

$$\begin{aligned} g(f(x)) &= g(\sqrt{x-5}) \\ &= (\sqrt{x-5})^2 \\ &= x-5. \end{aligned}$$

and it looks like the domain of $g(f(x))$ is the set of all real numbers. **But this is wrong.**

Instead you should do the following:

The number x is in the domain of $g(f(x))$ when

- (1) x is in the domain of $f(x)$, **and**
- (2) $f(x)$ is in the domain of g .

Now x is in the domain of $f(x)$ when

$$x-5 \geq 0$$

or when

$$x \geq 5.$$

So the domain of f is

$$D_f = \{x \mid x \geq 5\}.$$

The second condition, that $f(x)$ be in the domain of g does not give any additional restriction on x since the domain of g is the set of all real numbers. So the domain of $g(f(x))$ is

$$\{x \mid x \geq 5\}.$$

Problem 2. Find the domain of $f(f(x))$ for the function

$$f(x) = \frac{3x - 7}{6x + 5}.$$

Solution: The number x is in the domain of $f(f(x))$ when

(1) x is in the domain of $g(x)$

and

(2) $f(x)$ is in the domain of f .

Now x is in the domain of $f(x)$ when

$$6x + 5 \neq 0$$

or when

$$x \neq -5/6.$$

So the domain of $f(x)$ is

$$\{x | x \neq -5/6\}.$$

Then $f(x)$ is in the domain of f when

$$f(x) \neq -5/6$$

or when

$$\frac{3x - 7}{6x + 5} \neq -\frac{5}{6}.$$

Multiplying both sides by $6(6x + 5)$ gives

$$6(3x - 7) \neq 5(6x + 5)$$

or

$$18x - 42 \neq 30x + 25$$

So

$$x \neq -19/4.$$

Conclusion: The domain of $f(f(x))$ is

$$D_{f \circ f} = \{x | x \neq -5/6 \text{ and } x \neq -19/4\}.$$